Identification of Optimal Solutions for Second-order Conic Optimization

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1. Summary

- The notion of the optimal partition for second-order conic optimization.
- Iteration complexity of the identification of the optimal partition.
- The quadratic convergence of the Newton's method to identify a maximally complementary optimal solution.

2. Problem Description

Second-order conic optimization (SOCO) problem is defined as

$$(P) \quad \min\{c^T x : \ Ax = b, \ x \in \mathcal{L}_+^{\bar{n}}\},$$
$$(D) \quad \max\{b^T y : \ A^T y + s = c, \ s \in \mathcal{L}_+^{\bar{n}}\},$$

where $\bar{n} = \sum_{i=1}^p n_i$, $c \in \mathbb{R}^{\bar{n}}$, $A \in \mathbb{R}^{m \times \bar{n}}$, and

$$\mathcal{L}_+^{\bar{n}} := \mathbb{L}_+^{n_1} \times \mathbb{L}_+^{n_2} \times \cdots \times \mathbb{L}_+^{n_p},$$
$$\mathbb{L}_+^{n_i} := \{x^i = [x_1^i, x_2^i, \ldots, x_{n_i}^i]^T \in \mathbb{R}^{n_i} : x_1^i \geq \|x_{2:n_i}^i\|\}.$$

Assumption 1. *A has full row rank.*
Assumption 2. *Slater condition: $\exists$ feasible $(x,y,s) \mid x, s \in \operatorname{int}(\mathcal{L}_+^{\bar{n}})$.*
As a result, strong duality holds between (P) and (D).

3. Complementarity

The complementarity condition is written as $x \circ s = 0$, where

$$x^i \circ s^i := \begin{pmatrix} (x^i)^T s^i \\ x_1^i s_{2:n_i}^i + s_1^i x_{2:n_i}^i \end{pmatrix}, \qquad i = 1, \ldots, p.$$

Let $\mathcal{P}^*$ and $\mathcal{D}^*$ denote the primal and dual optimal sets, respectively.

- $(x^*, y^*, s^*) \in \mathcal{P}^* \times \mathcal{D}^*$ is **maximally complementary** if it has maximal number of second-order cones i for which
$$(x^i)^* + (s^i)^* \in \operatorname{int}(\mathbb{L}_+^{n_i}).$$
- $(x^*, y^*, s^*) \in \mathcal{P}^* \times \mathcal{D}^*$ is called **strictly complementary** if
$$x^* + s^* \in \operatorname{int}(\mathcal{L}_+^{\bar{n}}).$$

4. Central path equations

$$Ax = b, \quad x \in \operatorname{int}(\mathcal{L}_+^{\bar{n}}),$$
$$A^T y + s = c, \quad s \in \operatorname{int}(\mathcal{L}_+^{\bar{n}}),$$
$$x \circ s = \mu e.$$

- A unique solution $(x(\mu), y(\mu), s(\mu))$ so called a central solution $\forall \mu > 0$.
- As $\mu \to 0$, the trajectory converges to a maximally complementary optimal solution.

5. Optimal partition

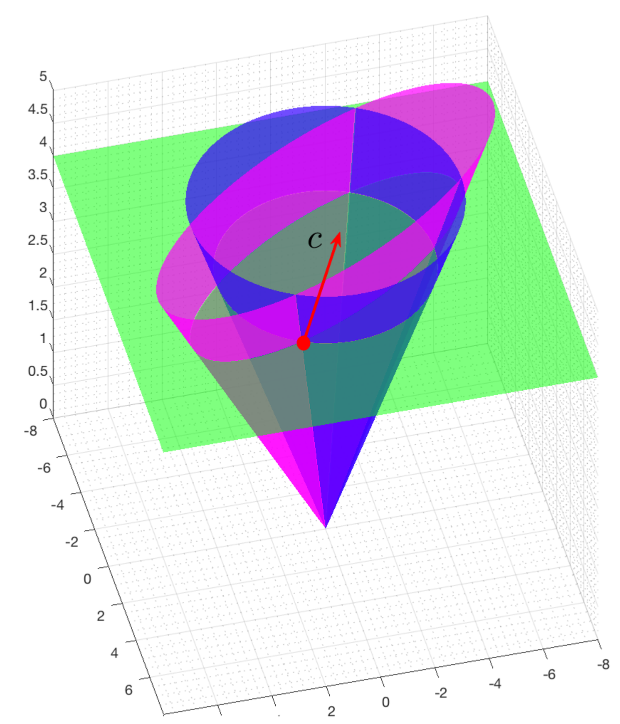
In SOCO, the index set $\{1, \cdots, p\}$ of the second-order cones is partitioned into four sets $\mathcal{B}, \mathcal{N}, \mathcal{R}$, and $\mathcal{T} = \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$ as follows.

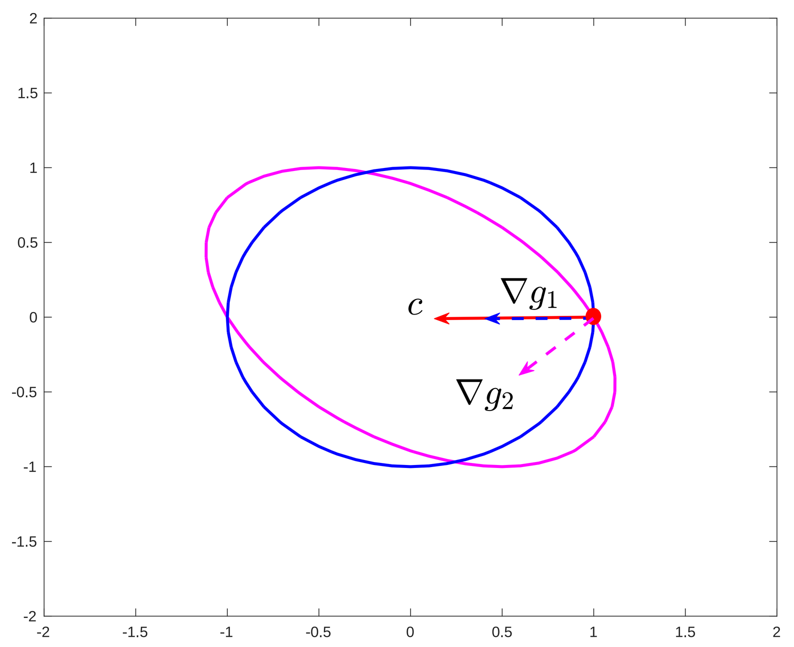
$$\mathcal{B} := \{i \mid x_1^i > \|x_{2:n_i}^i\|, \text{ for some } x \in \mathcal{P}^*\},$$
$$\mathcal{N} := \{i \mid s_1^i > \|s_{2:n_i}^i\|, \text{ for some } s \in \mathcal{D}^*\},$$
$$\mathcal{R} := \{i \mid x_1^i = \|x_{2:n_i}^i\| > 0, \ s_1^i = \|s_{2:n_i}^i\| > 0 \text{ for some } (x,y,s) \in \mathcal{P}^* \times \mathcal{D}^*\},$$
$$\mathcal{T}_1 := \{i \mid x^i = s^i = 0, \ \text{ for some } (x,y,s) \in \mathcal{P}^* \times \mathcal{D}^*\},$$
$$\mathcal{T}_2 := \{i \mid s^i = 0, \ x_1^i = \|x_{2:n_i}^i\| > 0, \text{ for some } (x,y,s) \in \mathcal{P}^* \times \mathcal{D}^*\},$$
$$\mathcal{T}_3 := \{i \mid x^i = 0, \ s_1^i = \|s_{2:n_i}^i\| > 0, \text{ for some } (x,y,s) \in \mathcal{P}^* \times \mathcal{D}^*\}.$$

- $(\mathcal{B}, \mathcal{N}, \mathcal{R}, \mathcal{T})$ is called the **optimal partition** of SOCO.

6. Example

- For this problem $\mathcal{R}, \mathcal{T}_2 \neq \emptyset$.
- The unique optimal solution is not strictly complementary.





7. Identification of $\mathcal{B}$, $\mathcal{N}$, $\mathcal{R}$, and $\mathcal{T}$

$$\sigma_{\mathcal{B}} := \min_{i \in \mathcal{B}} \max_{x \in \mathcal{P}^*} \{x_1^i - \|x_{2:n_i}^i\|\}, \qquad \sigma_{\mathcal{N}} := \min_{i \in \mathcal{N}} \max_{(y,s) \in \mathcal{D}^*} \{s_1^i - \|s_{2:n_i}^i\|\},$$
$$\sigma_1 := \min\{\sigma_{\mathcal{B}}, \sigma_{\mathcal{N}}\},$$
$$\sigma_2 := \min_{i \in \mathcal{R}} \max_{(x,y,s) \in \mathcal{P}^* \times \mathcal{D}^*} \{x_1^i + s_1^i - \|x_{2:n_i}^i + s_{2:n_i}^i\|\},$$
$$\sigma_3 := \max_{(x,y,s) \in \mathcal{P}^* \times \mathcal{D}^*} \{\|(x,y,s)\|\}.$$

Theorem 1. *We can identify $\mathcal{B}, \mathcal{N}, \mathcal{R}$ and $\mathcal{T}$ from a given central solution $(x(\mu), y(\mu), s(\mu))$ if*

$$\mu < \min\left\{\frac{\sigma_1^2}{2p^2}, \frac{\sigma_1 \sigma_2}{4p^2}, \frac{1}{p}\left(\frac{1}{3\kappa} \min\left\{\frac{\sigma_1}{2p}, \frac{\sigma_2}{2p}\right\}\right)^{\frac{1}{\gamma}}\right\},$$

where $\kappa > 0$ is a constant, $\gamma = 2^{-d_s}$ and d_s is the ***degree of singularity*** *of the optimal set.*

8. Primal and dual nondegeneracy

Let $T^s_{\mathcal{L}_+^{\bar{n}}}(x)$ $(T^s_{\mathcal{L}_+^{\bar{n}}}(s))$ be the tangent space to $\mathcal{L}_+^{\bar{n}}$ at x (s).

- A primal feasible solution x is called nondegenerate if
$$T^s_{\mathcal{L}_+^{\bar{n}}}(x) + \operatorname{Ker}(A) = \mathbb{R}^{\bar{n}}.$$
- A dual feasible solution (y, s) is called nondegenerate if
$$T^s_{\mathcal{L}_+^{\bar{n}}}(s) + \operatorname{Span}(A^T) = \mathbb{R}^{\bar{n}}.$$

9. Second-order sufficiency condition

The second-order sufficiency condition for (D) is written as

$$\sup_{x \in \mathcal{P}^*} \sum_{i=1}^p h^T \mathcal{H}^i(y,x) h > 0, \quad \forall h \in C(y) \setminus \{0\}, \tag{1}$$

where $C(y)$ denotes the cone of critical directions, and

$$\mathcal{H}^i(y,x) = \begin{cases} -\frac{x_1^i}{s_1^i} A_i R_i A_i^T, & s^i \in \operatorname{bd}(\mathbb{L}_+^{n_i}) \setminus \{0\}, \\ \mathbf{0}_{m \times m}, & \text{otherwise.} \end{cases}$$

Lemma 1. *Assume that the primal nondegeneracy condition holds. Then (1) holds at the unique solution $(y^*, s^*) \in \operatorname{rint}(\mathcal{D}^*)$.*

10. Nonlinear reformulation of (D)

Using the optimal partition we can obtain $(y^*, s^*) \in \operatorname{rint}(\mathcal{D}^*)$ by

$$\begin{array}{lll} \min & -b^T w & \\ \text{s.t.} & A_i^T w = c_i, & i \in \mathcal{B} \cup \mathcal{T}_1 \cup \mathcal{T}_2, \\ & A_i^T w + z^i = c_i, & i \in \mathcal{R} \cup \mathcal{T}_3, \\ & (z^i)^T R_i z^i = 0, & i \in \mathcal{R} \cup \mathcal{T}_3, \\ & (w, z) \in \mathcal{W}, & \end{array} \tag{2}$$

where $w \in \mathbb{R}^m$, $z^i \in \mathbb{R}^{n_i}$ for $i \in \mathcal{R} \cup \mathcal{T}_3$, and $\mathcal{W}$ is defined as

$$\mathcal{W} := \left\{(w,z) \mid z_1^i > 0, \ i \in \mathcal{R} \cup \mathcal{T}_3, \ \ c_i - A_i^T w \in \operatorname{int}(\mathbb{L}_+^{n_i}), \ i \in \mathcal{N}\right\}.$$

Lemma 2. *By the dual nondegeneracy condition, the set of Lagrange multipliers associated with the optimal solution of (2) is a singleton.*

11. Local convergence

Let β be an upper bound for the Jacobian of the system of optimality conditions at the unique optimal solution of (2), and define

$$\varepsilon := \frac{1}{4\beta \sqrt{\max\{\max_{i \in \mathcal{R} \cup \mathcal{T}_3}\{n_i\}, 2\}}}.$$

Theorem 2. *Assume that the primal-dual nondegeneracy conditions hold. Then, starting from $(x(\mu), y(\mu), s(\mu))$, where*

$$\mu \leq \left(\frac{\varepsilon^2}{\kappa^2 p^{2\gamma}\left(1 + \frac{1}{4}\left(\frac{4p\sqrt{|\mathcal{R}|}}{\sigma_2}\left(1 + \frac{2\sigma_3}{\sigma_2}\right)\right)^2\right)}\right)^{\frac{1}{2\gamma}},$$

the Newton's method applied to the KKT system of (2) converges to the unique optimal solution (y^*, s^*) with quadratic rate.